**Problem Set 7** due November 4, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

**Problem 1**: Consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

(1) Find a basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $\mathbb{R}^2$ such that $f(\boldsymbol{v}_1) = \boldsymbol{e}_2$ and $f(\boldsymbol{v}_2) = \boldsymbol{e}_3$, where $\boldsymbol{e}_i$ is the $i$–th coordinate unit vector. Compute the matrix $B$ which represents $f$ in the new basis, i.e.:

$$f(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2) = (b_{11}x_1 + b_{12}x_2)\boldsymbol{e}_1 + (b_{21}x_1 + b_{22}x_2)\boldsymbol{e}_2 + (b_{31}x_1 + b_{32}x_2)\boldsymbol{e}_3$$

and say explicitly how $B$ relates to $A$ (hint: $B$ should be equal to $A$ times a matrix). *(10 points)*

(2) Find a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ of $\mathbb{R}^3$ such that $f(\boldsymbol{e}_1) = \boldsymbol{w}_1$ and $f(\boldsymbol{e}_2) = \boldsymbol{w}_2$. Compute the matrix $C$ which represents $f$ in the new basis, i.e.:

$$f(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) = (c_{11}x_1 + c_{12}x_2)\boldsymbol{w}_1 + (c_{21}x_1 + c_{22}x_2)\boldsymbol{w}_2 + (c_{31}x_1 + c_{32}x_2)\boldsymbol{w}_3$$

and say explicitly how $C$ relates to $A$ (hint: $C$ should be equal to $A$ times a matrix). *(10 points)*

**Problem 2**: Let $A_n$ be the $n \times n$ matrix with 2's on the diagonal, and $-1$'s directly above and directly below. For example:

$$A_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(1) Compute $\det A_2$, $\det A_3$, $\det A_4$ by row operations (i.e. putting the matrix in question in row echelon form, multiplying the pivots then multiplying by $(-1)^{\#\text{ of row exchanges performed}}$). *(10 points)*

(2) Use cofactor expansion along the first row to obtain a recursive formula for $\det A_n$ in terms of $\det A_{n-1}$ and $\det A_{n-2}$, for all natural numbers $n$. *(10 points)*

(3) Guess what $\det A_n$ is in general, and use the recursive formula in part (2) to prove your guess. *(5 points)*

**Problem 3**: Let $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ be the $3 \times 3$ matrix defined in the previous problem.

(1) Use the cofactor formula for the inverse to compute $A_3^{-1}$. *(10 points)*

(2) Use Cramer's rule to compute the solution to the equation:

$$A_3 \boldsymbol{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for any numbers $b_1$, $b_2$, $b_3$. *(5 points)*

**Problem 4**: Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be any two vectors in $\mathbb{R}^n$ which are <u>not</u> orthogonal.

(1) What is the rank of the matrix $A = \boldsymbol{v}\boldsymbol{w}^T$? *(5 points)*

(2) Show that $\boldsymbol{v}$ is an eigenvector of the matrix $A$. What is the corresponding eigenvalue? *(5 points)*

(3) Find $n-1$ other eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}$, which together with $\boldsymbol{v}$ form a basis of $\mathbb{R}^n$ (your choice will depend on $\boldsymbol{w}$). What are the corresponding eigenvalues of these $n-1$ eigenvectors? *(10 pts)*

**Problem 5**: Consider the following block matrix (where $A, B, C$ are $2 \times 2$ blocks):

$$X = \left[\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 2 & 0 & 3 \\ -3 & 5 & 9 & 0 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

(1) Compute the eigenvalues and eigenvectors of $A$ and $B$. *(10 points)*

(2) Compute the characteristic polynomial and the eigenvalues of $X$ (Hint: remember part (1)). What is the relationship between $\det X$, $\det A$, and $\det B$? *(10 points)*